

ON A FROBENIUS PROBLEM FOR POLYNOMIALS

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ABSTRACT. We extend the famous diophantine Frobenius problem to the case of polynomials over a field k . Similar to the classical problem, we show that the $n = 2$ case of the Frobenius problem for polynomials is easy to solve. In addition, we translate a few results from the Frobenius problem over \mathbb{Z} to $k[t]$ and give an algorithm to solve the Frobenius problem for polynomials over a field k of sufficiently large size.

1. INTRODUCTION

The Frobenius problem (FP) is a problem in basic number theory related to nonnegative integer solutions (x_1, \dots, x_n) of

$$x_1 a_1 + \dots + x_n a_n = f,$$

where the a_i 's and f are positive integers and $\gcd(a_1, \dots, a_n) = 1$. In particular, the Frobenius number $g = g(a_1, \dots, a_n)$ is the largest f so that this equation fails to have a solution and the Frobenius problem is to compute g . This classical problem has a long history and has found many applications in mathematics as seen in the book [RA05], which contains the state of the art on FP as well as almost 500 references on the subject and its applications.

As early as the mid-nineteenth century, mathematicians started to notice a strong relationship between the ring of integers \mathbb{Z} and the ring of polynomials $k[t]$ over a field k , specially when k is finite. The discovery of this connection has proved very fruitful to number theory and it has grown into an area of active research known as the arithmetic of function fields (see for instance [Ros02, Tha04]). In the arithmetic of function fields, many of the classical results and conjectures in number theory (such as the Prime Number Theorem, Falting's Theorem, and the Riemann Hypothesis to name a few) have found an analogous statement over $k[t]$. Surprisingly, FP is one of the few classical and folkloric results in number theory for which an analogous statement over function fields cannot be found in the literature. The main goal of this note is to propose an analogous FP over $k[t]$.

The first thing to notice is that unlike the classical case where every non-zero integer is either positive or negative, we have many different ways of choosing the “sign” of a polynomial since the set of units of $k[t]$ is k^* . Nonetheless, a polynomial that is either the zero polynomial or monic is a natural choice for the notion of a “non-negative” polynomial.

Definition 1.1. We will denote by $k[t]_{\geq 0}$ the set of all monic polynomials over a field k together with the zero element.

Given monic polynomials A_1, \dots, A_n, F , our formulation of FP over $k[t]$ is related to solutions of

$$(1.1) \quad x_1 A_1 + \dots + x_n A_n = F,$$

with $x_i \in k[t]_{\geq 0}$. It is based on the following theorem, whose prove we delay until the next section.

Theorem 1.2. *Let $n \geq 2$ be an integer and let A_1, \dots, A_n be coprime monic polynomials in $k[t]$. Then there exists an integer $g = g(A_1, \dots, A_n)$ such that for all monic polynomial F with $\deg F > g$ there exists a solution to (1.1) with $x_1, \dots, x_n \in k[t]_{\geq 0}$.*

Based on this result, below we give a statement over $k[t]$ that is analogous to the classical Frobenius problem.

Definition 1.3. If (1.1) has a solution in $k[t]_{\geq 0}$ for all monic polynomial F then we define $g(A_1, A_2, \dots, A_n) = -\infty$. Otherwise, we define $g(A_1, \dots, A_n)$ as the largest degree of a monic polynomial F for which equation (1.1) has no solutions in $k[t]_{\geq 0}$. We call $g(A_1, \dots, A_n)$ the *frobenius degree* of A_1, \dots, A_n .

The Frobenius Problem for Polynomials in dimension n – FPP. *Given coprime monic polynomials A_1, \dots, A_n , compute $g(A_1, \dots, A_n)$.*

Remark 1.4. It is worth noting that, technically, $g = g(A_1, \dots, A_n)$ also depends on the field k over which the A_i 's are defined. There are two reasons why we have dropped from the notation of g the dependence on the base field. First, we will be mostly concerned on computing g over a fixed field k . Second, although there are instances where g changes if we replace k by one of its extension field K , it turns out that g is not affected by field extensions as long as $|k|$ is sufficiently large. See Section 3.2 and Corollary 6.10 for a proof of a more precise version of this statement.

Generally, when comparing \mathbb{Z} and $k[t]$, the role of the absolute value function in $k[t]$ is played by the degree of a polynomial. But unlike \mathbb{Z} , $k[t]$ does not satisfy the well-ordering principle, and a set of polynomials of bounded “size” does not contain a “largest” polynomial. Therefore, there is not a unique polynomial F with $\deg F = g(A_1, \dots, A_n)$ for which (1.1) has no solution in $k[t]_{\geq 0}$. This observation inspires the following definition.

Definition 1.5. If $g = g(A_1, \dots, A_n) > -\infty$, then a monic polynomial F with $\deg F = g$ for which equation (1.1) has no solutions in $k[t]_{\geq 0}$ is said to be a *counter-example to FPP* for A_1, \dots, A_n .

In light of this definition, an alternative version of **FPP** could deal with not only the computation of $g(A_1, \dots, A_n)$, but also with the construction of a counter-example to **FPP**. In Section 6, we provide an algorithm that in most cases solve both versions of **FPP**. It is worth pointing out that

constructing a counter-example to **FPP** seems to be more computationally challenging than simply finding $g(A_1, \dots, A_n)$.

The rest of this article is dedicated to further comparison between the classical FP and **FPP**, and it is organized as follows. In the next section, we give a proof of Theorem 1.2. In Section 3 we make some remarks on **FPP** and how it differs from the classical problem. Section 4 is devoted to presenting two examples for which the Frobenius degree can be computed explicitly. These examples are used to solve **FPP** for dimension 2 and to prove that the upper and lower bounds given in the text for $g(A_1, \dots, A_n)$ are sharp. Section 5 gives a version for polynomials of the classical denumerant function and compute an asymptotic formula in dimension 2 that resembles Schur's classical asymptotic formula for FP. In the last section, we give an algorithm to solve **FPP** for $n \geq 3$, and prove that $g(A_1, \dots, A_n)$ is not affected by base field extensions K/k , if $|k|$ is sufficiently large.

2. PROOF OF THEOREM 1.2

The proof of Theorem 1.2 is by induction on n . In the following lemma, we prove the base case for induction.

Lemma 2.1. *Let A and B be coprime monic polynomials in $k[x]$. If $F \in k[t]_{\geq 0}$ and $\deg F > \deg A + \deg B$, then there exist $x, y \in k[t]_{\geq 0}$ such that $F = xA + yB$.*

Proof. If (x_0, y_0) is a particular solution of the linear equation $F = Ax + By$, then its general solution is given by $x = x_0 + uB$ and $y = y_0 - uA$, where u is an arbitrary polynomial. This implies that we can write $F = x_0A + y_0B$ with $\deg x_0 < \deg B$. Since $\deg F > \deg A + \deg B$, we conclude that $y_0B = F - x_0A$ is a monic polynomial of degree $> \deg A + \deg B$. As a consequence, we have that y_0 is monic and $\deg y_0 > \deg A$. Let $x = x_0 + B$ and $y = y_0 - A$. Then x and y lie on $k[t]_{\geq 0}$ and $F = xA + yB$. \square

Remark 2.2. (1) If one follows the proof of Lemma 2.1, one can actually show that if $\deg F > \deg A + \deg B$, then there exists a solution in $k[t]_{\geq 0}$ to $Ax + By = F$ that satisfies $\deg x = \deg B$ and $\deg y = \deg F - \deg B$. The argument is symmetric in A and B , and guarantees the existence of solutions satisfying $\deg x = \deg F - \deg A$ and $\deg y = \deg A$.

(2) Notice that this proves that $g(A, B) \leq \deg A + \deg B$.

Proof of Theorem 1.2. We remind the reader that the proof is by induction on n . If $\gcd(A_1, \dots, A_{n-1}) = 1$, then the result follows by induction. Thus we assume that $\gcd(A_1, \dots, A_{n-1}) = D$ with D a monic polynomial of positive degree. Write $\tilde{A}_i = A_i/D$. Notice that $\gcd(\tilde{A}_1, \dots, \tilde{A}_{n-1}) = 1$ and $\gcd(A_n, D) = 1$. By the induction hypothesis, there exists an integer $\tilde{g} = g(\tilde{A}_1, \dots, \tilde{A}_{n-1})$ such that the equation $x_1\tilde{A}_1 + \dots + x_{n-1}\tilde{A}_{n-1} = z$ has a solution satisfying $x_1, \dots, x_{n-1} \in k[t]_{\geq 0}$ whenever $\deg z > \tilde{g}$. We will

prove that (1.1) has a solution with $x_1, \dots, x_n \in k[t]_{\geq 0}$ whenever

$$(2.1) \quad \deg F > \max\{\deg A_n, \tilde{g}\} + \deg D.$$

First notice that (2.1), Lemma 2.1 and Remark 2.2 imply that the equation

$$x_n A_n + z D = F$$

has a solution with $x_n, z \in k[t]_{\geq 0}$ and $\deg z = \deg F - \deg D$. This together with (2.1) imply $\deg z > \tilde{g}$. Therefore, by the induction hypothesis, the equation

$$x_1 \tilde{A}_1 + \dots + x_n \tilde{A}_{n-1} = z = \frac{F - x_n A_n}{D},$$

has a solution with $x_1, \dots, x_{n-1} \in k[t]_{\geq 0}$ and the result follows after multiplying the last equation by D . \square

Remark 2.3. Notice that implicit in the proof of Theorem 1.2 are the following upper bounds for the Frobenius degree of coprime monic polynomials A_1, \dots, A_n with $n > 2$. If $\gcd(A_1, \dots, A_{n-1}) = 1$ then

$$g(A_1, \dots, A_n) \leq g(A_1, \dots, A_{n-1}).$$

If $D = \gcd(A_1, \dots, A_{n-1})$ has positive degree then

$$g(A_1, \dots, A_n) \leq \max\{\deg A_n, g(A_1/D, \dots, A_{n-1}/D)\} + \deg D.$$

Remark 2.4. Clearly, the upper bound given in the previous remark depends on the ordering of the A_i 's and the computation of the Frobenius degree of $n - 1$ coprime polynomials. To avoid such dependence, we consider $S = \{B_1, \dots, B_m\}$ to be a subset of $\{A_1, \dots, A_n\}$, and define inductively the following function $U(S)$. We let $U(S) = \deg B_1 + \deg B_2$, if $m = 2$. Otherwise, $U(S) = U(B_1, \dots, B_{m-1})$, if $\gcd(B_1, \dots, B_{m-1}) = 1$; or $D_S = \gcd(B_1, \dots, B_{m-1})$ has positive degree and

$$U(S) = \max\{\deg B_m, U(B_1/D_S, \dots, B_{m-1}/D_S)\} + \deg D_S.$$

Thus Remark 2.3 and Lemma 2.1 implies that for $n > 2$

$$g(A_1, \dots, A_n) \leq \min\{U(S) : S \subset \{1, \dots, n\}, |S| = n - 1\}.$$

3. REMARKS ON FPP

As we noted in the introduction, unlike \mathbb{Z} , the units in the ring of polynomials $k[t]$ can be quite large. Although this difference allows one to be flexible when choosing the “sign” of a polynomial, it does not prevent FPP to be a well posed problem in the arithmetic of function fields. We do have a few others significant difference between \mathbb{Z} and $k[t]$ which creates some striking differences between the classical FP and FPP. In this section we present two results that stem from these differences and which are intrinsic to the function field setting.

The first notable difference is given by the existence of base fields of positive characteristic p . As we show in Theorem 3.1 below, FPP in dimension n is easy to solve if $n \geq p$.

As noted in the introduction, another striking difference between the classical and the polynomial Frobenius problem is the existence of base field extensions of the ring $k[t]$. We show in this section that if we fix coprime monic polynomials A_1, \dots, A_n over $k[t]$, then for some field extension K/k , $g(A_1, \dots, A_n)$ may increase if we consider solutions of (1.1) over $K[t]_{\geq 0}$ instead.

3.1. Issues in positive characteristic. When comparing the arithmetic of \mathbb{Z} and $k[t]$, it is often the case that the analogy is tighter if we take k to be a finite field. This is also the case for FPP, since for k finite with characteristic $p < n$, we are able to show that (1.1) has a finite number of solutions in $k[t]_{\geq 0}$, in perfect analogy with FP over \mathbb{Z} .

Let k be a field of characteristic $p \geq 0$. If $n < p$ or $p = 0$ then

$$\deg F = \max\{\deg A_i + \deg x_i : 1 \leq i \leq n\}.$$

This implies that for all $1 \leq i \leq n$,

$$(3.1) \quad \deg x_i \leq \deg F - \min_{1 \leq i \leq n} \{\deg A_i\}.$$

In particular, whenever $p = 0$ or $n < p$, the polynomials x_i 's in a solution of (1.1) have bounded degree. As we show below, the condition $n < p$ or $p = 0$ is not only sufficient but also necessary in order for the monic solutions to the equation (1.1) to have bounded degree. Since the set of polynomials of bounded degree is finite when k is finite, this result allows us to give a criteria for when (1.1) has a finite number of solutions in $k[t]_{\geq 0}$.

Theorem 3.1. *Let A_1, A_2, \dots, A_n be coprime monic polynomials in $k[t]$, with k a field of characteristic $p > 0$. For any monic polynomial F , the equation (1.1) has solutions $x_i \in k[t]_{\geq 0}$ with arbitrarily large degree if and only if $n \geq p$.*

Proof. As discussed above, if $n < p$ then the degrees of the solutions $x_i \in k[t]_{\geq 0}$ of (1.1) are bounded above by (3.1). Thus we are left to show that if $n \geq p$ then (1.1) has solutions $x_i \in k[t]_{\geq 0}$ of unbounded degree. Write $n = ap + b$ with $a > 0$ and $0 \leq b < p$. We first consider the case where $b \neq 0$.

Let $R = \{1, 2, \dots, pa\}$ and $S = \{n - p + 1, n - p + 2, \dots, n\}$. Notice that $R \cup S = \{1, 2, \dots, n\}$, $|S|$ and $|R|$ are divisible by p and that $|R \cap S| = p - b + 1$. For $s \in S$ and $r \in R$, the monic polynomials $y_s = (\prod_{l \in S} A_l)/A_s$ and $z_r = (\prod_{l \in R} A_l)/A_r$ satisfy

$$(3.2) \quad \sum_{s \in S} y_s A_s = \sum_{r \in R} z_r A_r = 0.$$

Since $\gcd(A_1, \dots, A_n) = 1$, we can find polynomials G_1, \dots, G_n such that $F = A_1 G_1 + \dots + A_n G_n$. Let l and m be positive integers satisfying

$$l > m + \max\{\deg z_r : r \in R\} > \max\{\deg G_i : 1 \leq i \leq n\}.$$

Thus the polynomials

$$x_i = \begin{cases} t^l y_i + G_i, & \text{if } i \in R \setminus R \cap S \\ t^l y_i + t^m z_i + G_i, & \text{if } i \in R \cap S \\ t^m z_i + G_i, & \text{if } i \in S \setminus R \cap S \end{cases}$$

are monic and have unbounded degree. The result follows from (3.2) and the following computation

$$\begin{aligned} \sum_{i=1}^n x_i A_i &= \sum_{i \in R \setminus R \cap S} (t^l y_i + G_i) A_i + \sum_{i \in R \cap S} (t^l y_i + t^m z_i + G_i) A_i + \sum_{i \in S \setminus R \cap S} (t^m z_i + G_i) A_i \\ &= t^l \sum_{i \in R} y_i A_i + t^m \sum_{i \in S} z_i A_i + \sum_{i=1}^n G_i A_i = F \end{aligned}$$

After some minor adjustments, the above proof works for $b = 0$ if we regard $S = \emptyset$. \square

Remark 3.2. The previous result also shows that over a field of positive characteristic p , $g(A_1, \dots, A_n) > -\infty$ if and only if $n < p$ or $1 \notin \{A_1, \dots, A_n\}$. While in the classical case we have $g(A_1, \dots, A_n) > -\infty$ if and only if $1 \notin \{A_1, \dots, A_n\}$.

3.2. FPP over extensions of the base field. Another critical difference between the arithmetic of function fields and that of \mathbb{Q} is the existence of constant field extensions. Concerning FPP, we first observe that, for a fixed set of coprime monic polynomials A_1, \dots, A_n over k , our definition of the Frobenius degree is, a priori, dependent on the base field k . In order to study such dependence on the base field, given a field extension K/k , we write $g_K = g_K(A_1, \dots, A_n)$ for the largest degree of a monic polynomial F over K for which (1.1) has no solutions in $K[t]_{\geq 0}$. Clearly, $g_k \leq g_K$ for any field extension K/k . As we show below, there are examples of field extensions K/k where $g_k < g_K$.

Example 3.3. Let $A_i = t + i$. Remark 2.4 implies that $g(A_1, A_2, A_3) \leq 2$. To find all monic polynomials F of degree 2 for which (1.1) has a solution in $k[t]_{\geq 0}$, we only need to compute all possible linear combinations

$$(3.3) \quad x(t+1) + y(t+2) + z(t+3),$$

with $(x, y, z) \in (k[t]_{\geq 0})^3$ and $\deg x = 1$ and $\deg y, \deg z < 1$; or $\deg y = 1$ and $\deg x, \deg z < 1$; or $\deg z = 1$ and $\deg x, \deg y < 1$.

If we take $k = \mathbb{F}_5$, then a computer search shows that all degree 2 monic polynomials appear as the linear combination described in (3.3). This shows that $g_k(A_1, A_2, A_3) < 2$. On the other hand, the same computation with $K = \mathbb{F}_{5^2}$ shows that not all degree 2 polynomials appear as a linear combination in (3.3); hence $g_K(A_1, A_2, A_3) = 2$.

In Section 6, we extend the basic idea described in the previous example of looking at all possible monic linear combinations of A_1, \dots, A_n . We

use it to prove that $g_k = g_K$, if $|k|$ is “sufficiently large”. In particular, $g_K(A_1, \dots, A_n)$ is independent of the field extension K/k , whenever k is infinite. The proof is given in Corollary 6.10, where we also describe how large $|k|$ needs to be in order to ensure that $g_K = g_k$.

4. TWO INTERESTING EXAMPLES

In this section we give two examples of families of coprime monic polynomials A_1, \dots, A_n for which we can compute $g(A_1, \dots, A_n)$ explicitly. Later, such examples will be used to prove that the upper and lower bounds given by Remark 2.3 and Corollary 6.8, respectively, are sharp. Additionally, we use the result below to settle the two dimensional case of **FPP**.

Lemma 4.1. *Let A_1, \dots, A_n be pairwise coprime monic polynomials over a field k . Suppose $\text{char}(k) = 0$ or $n < \text{char}(k)$. Define $P = \prod_{i=1}^n A_i$, $\tilde{A}_i = P/A_i$ and $F = P - \sum_{i=1}^n \tilde{A}_i$.*

Then the equation

$$x_1 \tilde{A}_1 + \dots + x_n \tilde{A}_n = F,$$

has no solution with $x_i \in k[t]_{\geq 0}$. Moreover,

$$g(\tilde{A}_1, \dots, \tilde{A}_n) = \deg F = \deg A_1 + \dots + \deg A_n.$$

Proof. Suppose, for the sake of contradiction, that we can find $x_i \in k[t]_{\geq 0}$ satisfying $x_1 \tilde{A}_1 + \dots + x_n \tilde{A}_n = P - \sum_{i=1}^n \tilde{A}_i$. This implies that

$$(4.1) \quad (x_1 + 1) \tilde{A}_1 + \dots + (x_n + 1) \tilde{A}_n = \prod_{i=1}^n A_i,$$

and that $A_i \mid \tilde{A}_i(x_i + 1)$. Since by hypothesis $\gcd(A_i, \tilde{A}_i) = 1$, we have that

$$(4.2) \quad x_i + 1 = A_i B_i$$

for some $B_i \in k[t]_{\geq 0}$. From (4.2) and (4.1), we arrive at

$$B_1 + \dots + B_n = 1.$$

The hypothesis on $\text{char}(k)$ and the fact that B_1, \dots, B_n are monic imply that $0 = \deg 1 = \max\{\deg B_1, \dots, \deg B_n\} \geq \deg B_i$ and, consequently, $B_i = 0$ for all but one $i \in \{1, \dots, n\}$; say $i = 1$. Therefore (4.2) implies that $x_i = -1$, for $i \neq 1$. This contradicts the fact that $x_i \in k[t]_{\geq 0}$ and the result follows.

To prove the “moreover” part we first note that the argument above proves that $g(\tilde{A}_1, \dots, \tilde{A}_n) \geq \deg F$. To finish the proof we show by induction on n that if A_1, \dots, A_n are pairwise coprime monic polynomials then $g(\tilde{A}_1, \dots, \tilde{A}_n) \leq \deg A_1 + \dots + \deg A_n$.

The base case $n = 2$ was proved in Lemma 2.1. Let $P_{n-1} = \prod_{i=1}^{n-1} A_i$ and $\tilde{A}_{i,n-1} = P_{n-1}/A_i$, for $1 \leq i \leq n-1$. By the induction hypothesis

$$g(\tilde{A}_{1,n-1}, \dots, \tilde{A}_{n-1,n-1}) \leq \deg A_1 + \dots + \deg A_{n-1}.$$

Notice that $\tilde{A}_{i,n-1} = \tilde{A}_i/A_n$ and that $A_n = \gcd(\tilde{A}_1, \dots, \tilde{A}_{n-1})$. This fact and the upper bound in Remark 2.3 imply that

$$\begin{aligned} g(\tilde{A}_1, \dots, \tilde{A}_n) &\leq \max \left\{ \deg \tilde{A}_n, g \left(\tilde{A}_1/A_n, \dots, \tilde{A}_{n-1}/A_n \right) \right\} + \deg A_n \\ &\leq \max \left\{ \sum_{i=1}^{n-1} \deg A_i, g \left(\tilde{A}_{1,n-1}, \dots, \tilde{A}_{n-1,n-1} \right) \right\} + \deg A_n \\ &\leq \deg A_1 + \dots + \deg A_n, \end{aligned}$$

as desired. \square

Clearly, the two-dimensional case of **FPP** is the case $n = 2$ of the previous result. Still, we restate it below for future reference.

Corollary 4.2. *Let A and B be coprime monic polynomials over a field k . Suppose $\text{char}(k) = 0$ or $\text{char}(k)$ is odd. Then*

$$g(A, B) = \deg A + \deg B,$$

and $G = AB - A - B$ is a counter-example to **FPP** for A, B .

Remark 4.3. It is enlightening to compare this with the classical Frobenius problem. In the latter case, Sylvester's celebrated result shows that $g(p, q) = pq - p - q$ for relatively prime positive integers p and q . As we saw above, the natural translation of this formula over to $k[t]$ solves **FPP** in dimension 2.

The next result shows that for all $n \geq 2$ the upper bound in Remark 2.3 cannot be improved.

Lemma 4.4. *Let A_1, A_2, \dots, A_n be coprime non-constant monic polynomials over a field k . Suppose $\text{char}(k) = 0$ or $n < \text{char}(k)$. Suppose $\gcd(A_1, \dots, A_{n-1}) = D$ and $D \neq A_i$ for all $1 \leq i \leq n-1$. If $\deg A_n > g(A_1/D, \dots, A_{n-1}/D)$ then*

$$g(A_1, \dots, A_n) = \max\{\deg A_n, g(A_1/D, \dots, A_{n-1}/D)\} + \deg D,$$

or

$$g(A_1, \dots, A_n) = g(A_1, \dots, A_{n-1}),$$

if $D \neq 1$ or $D = 1$, respectively.

Proof. We assume that $D \neq 1$, since the case $D = 1$ is simpler and can be proved in a similar way. Remark 2.3 provides us with the upper bound

$$g(A_1, \dots, A_n) \leq \max\{\deg A_n, g(A_1/D, \dots, A_{n-1}/D)\} + \deg D.$$

Let us show that equality above holds by constructing a counter-example of **FPP** for A_1, \dots, A_n with the appropriate degree.

We write $\tilde{g} = g(A_1/D, \dots, A_{n-1}/D)$. Let G with $\deg G = \tilde{g}$ be a counterexample for the Frobenius problem for $A_1/D, \dots, A_{n-1}/D$ (which exists since $A_i/D \neq 1$ for all i). We will show that the equality

$$(4.3) \quad x_1 A_1 + \dots + x_n A_n = DG + (D-1)A_n$$

does not hold with $x_1, \dots, x_n \in k[t]_{\geq 0}$. Assume that the opposite happens. Since by hypothesis $\deg A_n > \tilde{g}$, comparison of degrees in (4.3) yields $\deg x_n \leq \deg D$. This fact together with $\gcd(D, A_n) = 1$ and

$$D \left(x_1 \frac{A_1}{D} + \dots + x_{n-1} \frac{A_{n-1}}{D} - G \right) = (D - 1 - x_n) A_n$$

imply that $x_n = D - 1$. Consequently,

$$x_1 \frac{A_1}{D} + \dots + x_{n-1} \frac{A_{n-1}}{D} = G,$$

which contradicts the fact that G is a counter-example of **FPP** for the polynomials $A_1/D, \dots, A_{n-1}/D$. Since

$$\deg(DG + (D - 1)A_n) = \max\{\deg A_n, g(A_1/D, \dots, A_{n-1}/D)\} + \deg D,$$

the result follows. \square

5. THE TYPE-DENUMERANT FUNCTION

The following is a natural problem closely related to the classical FP:

- Given a positive integer f find the number $d(f; a_1, \dots, a_n)$ of solutions of $x_1 a_1 + \dots + x_n a_n = f$ with integers $x_i \geq 0$.¹

In this section we provide an analogous statement in the 2-dimensional case to the following classical result associated to the above problem.

Theorem 5.1 (Schur, see Theorem 4.2.1 in [RA05]). *Let a_1, \dots, a_n be coprime positive integers and let $P_n = \prod_{i=1}^n a_i$. Then,*

$$d(f; a_1, \dots, a_n) \sim \frac{f^{n-1}}{P_n(n-1)!}, \text{ as } m \rightarrow \infty.$$

Before translating such result to **FPP**, we make the following observation. Cf. Theorem 3.1, for a fixed monic polynomial F , (1.1) has an infinite number of solutions with $x_i \in k[t]_{\geq 0}$; unless k is finite and $\text{char}(k) > n$. We circumvent this difficulty through the following definition.

Definition 5.2. Let A_1, \dots, A_n be coprime monic polynomials. For a fixed monic polynomial F , we define the *type of a solution* $\mathbf{x} = (x_1, \dots, x_n) \in (k[t]_{\geq 0})^n$ of (1.1) as the n -tuple $T(\mathbf{x}) = (\deg x_1, \dots, \deg x_n)$. The number of types associated to F is given by the *type-denumerant* function $\mathcal{T}(F; A_1, \dots, A_n)$.

Remark 5.3. Over a field k of characteristic p , (3.1) and Theorem 3.1 imply that $\mathcal{T}(F; A_1, \dots, A_n)$ is finite if and only if $n < p$ or $p = 0$. Thus, if $p \leq n$, $\mathcal{T}(F; A_1, \dots, A_n)$ is still not analogous to the classical denumerant function.

¹The function $d(f; a_1, \dots, a_n)$ is known as the *denumerant* function.

A natural guess for an analogous statement over $k[t]$ of Theorem 5.1 can be obtained if we replace f, a_1, \dots, a_n by polynomials and the right hand-side of the asymptotic formula by an expression involving the degrees of such polynomials, as follows:

$$\mathcal{T}(F; A_1, \dots, A_n) \sim (n-1) \deg F - \sum_{i=1}^n \deg A_i, \quad \text{as } \deg F \longrightarrow \infty.$$

As we show below, this formula is invalid already for $n = 2$. Nonetheless, our asymptotic formula for $\mathcal{T}(F; A_1, A_2)$ is still in line with what one would expect by considering Theorem 5.1 as a starting point. It is still unclear what an asymptotic value for $\mathcal{T}(F; A_1, \dots, A_n)$ should be.

Theorem 5.4. *Let A and B be coprime monic polynomials over a field of zero or odd characteristic. Then, for some integer $0 \leq C \leq 2$,*

$$\mathcal{T}(F; A, B) = 2(\deg F - \deg A - \deg B) + C,$$

if $\deg F > A + B$. In particular,

$$\mathcal{T}(F; A, B) \sim 2 \deg F - \deg A - \deg B, \quad \text{as } \deg F \longrightarrow \infty.$$

Proof. We let $f = \deg F$, $a = \deg A$ and $b = \deg B$. Assume first that $f = \deg x + a$. Then, the possible types in this case are of the form $(f - a, c)$, with $c = -\infty$ or $0 \leq c < f - b$. If $f > a + b$, then by Remark 2.2 we know there is a solution of type $(f - a, a)$, say (x_0, y_0) . Let z be some monic polynomial with $1 \leq \deg z < f - a - b$. Then, $(x_0 - zB, y_0 + zA)$ is a solution of type $(f - a, \deg z + a)$. In this fashion we can obtain all types of the form $(f - a, c)$ with $a \leq c < f - b$. This leaves out solutions of type $(f - a, m)$ with $m < a$. Assume (x', y') is another solution of $xA + yB = F$. Then, $(x - x')A = (y' - y)B$ and $y' \equiv y \pmod{A}$. If $\deg y, \deg y' < a$ then $y = y'$. Thus, there is at most one solution of type $(f - a, m)$ with $m < a$. We then have: if $f > a + b$ then the number of types of solutions with type $(f - a, c)$ is $f - a - b + \chi_{(A,B)}(F)$, where $\chi_{(A,B)}(F) \in \{0, 1\}$ is the number of types $(f - a, c)$ with $c < a$. The above argument applies to solutions of type $(c, f - b)$ as well, and the result follows. \square

Example 5.5. Let $\chi_{(A,B)}(F) \in \{0, 1\}$ be the number of types $(f - a, c)$ with $c < a$ and x be a monic polynomial with $\deg x \geq 1$. Then,

- If $F = xAB$ then $\chi_{(A,B)} = \chi_{(B,A)} = 1$ since we have types of the form $(f - a, -\infty)$ and $(-\infty, f - b)$.
- If $F = xBA + (A + 2)B$, then, $\chi_{(A,B)} = 0$ since 2 is the only polynomial of degree $< a$ congruent to $(A + 2) \pmod{A}$. $\chi_{(B,A)}$ is obviously 1.
- If $F = (xB + 2)A + (A + 2)B$ then $\chi_{(A,B)} = \chi_{(B,A)} = 0$.

Looking at the examples above, we can see that $\chi_{(A,B)} = 1$ if, and only if, for any solution (x, y) of type $(f - a, c)$, we have that the representative of $y \pmod{A}$ of degree less than a is in $k[t]_{\geq 0}$.

6. AN ALGORITHM TO SOLVE **FPP**

6.1. Set-up and notation. In this section we construct an algorithm to compute $g(A_1, \dots, A_n)$. To avoid trivial cases, we assume $1 \notin \{A_1, \dots, A_n\}$ and that the characteristic of the base field is either 0 or greater than n . Our algorithm to solve **FPP** is based on the fact that given a fixed polynomial F , one can decide whether or not (1.1) has a solution (x_1, \dots, x_n) by solving a system of linear equations that is obtained from considering the coefficients of the polynomials in (x_1, \dots, x_n) as variables. In order to make the above idea more precise, we use the following notation:

- We write $a_i = \deg A_i$ and

$$A_i = t^{a_i} + \sum_{j=0}^{a_i-1} \alpha_{ij} t^j.$$

- d is a positive integer satisfying $d \geq \min\{a_i : 1 \leq i \leq n\}$.
- \mathcal{P}_d is the k -vector space of polynomials of degree $\leq d$.
- In k^{d+1} , we identify a polynomial $\sum_{k=0}^e \psi_i t^i \in \mathcal{P}_d$ of degree e with either the vector

$$(6.1) \quad (\underbrace{0, \dots, 0}_{d-e}, \underbrace{\psi_e, \psi_{e-1}, \dots, \psi_0}_{e+1}),$$

or a column matrix, which is the transpose of the vector above. We note that often we use the polynomial and matrix representation of an element in \mathcal{P}_d interchangeably.

- If $A_i \in \mathcal{P}_d$, we let D_i be the column matrix associated to A_i under the above identification of \mathcal{P}_d with k^{d+1} .
- \mathcal{M}_d is the set of monic polynomials of degree d .
- \mathcal{F}_d is the set of monic polynomials F of degree d for which (1.1) has a solution with $x_i \in k[t]_{\geq 0}$. Notice that $d = g(A_1, \dots, A_n)$ is the largest integer d for which $\mathcal{F}_d \subsetneq \mathcal{M}_d$.
- $\mathcal{T}_d = \mathcal{T}_d(a_1, \dots, a_n)$ is the set of n -tuples (e_1, \dots, e_n) such that:
 - (1) there exists a unique integer $j = j(T)$ such that $1 \leq j \leq n$ and $e_j = d - a_j \geq 0$; and
 - (2) for $i \neq j$, we have $e_i = -\infty$ or e_i is an integer satisfying $0 \leq e_i < d - a_i$.
- We consider the following subsets of the integers $1 \leq i \leq n$:

$$(6.2) \quad \mathcal{R}_T = \{i : 0 < e_i \leq d - a_i\} \quad \text{and} \quad \mathcal{S}_T = \{i : e_i = 0\}.$$

Notice that if $\mathcal{R}_T = \emptyset$ then $d = a_{j(T)}$, and consequently $\mathcal{S}_T \neq \emptyset$. Also, if $i \notin \mathcal{R}_T \cup \mathcal{S}_T$ then $e_i = -\infty$.

- We let the *index of T* to be $\text{ind}(T) = \sum_{i=1}^n \max(e_i, 0)$. Observe that $\mathcal{R}_T \neq \emptyset$ if and only if $\text{ind}(T) > 0$. If that is the case, then $\text{ind}(T) = \sum_{i \in \mathcal{R}_T} e_i$.

Remark 6.1. The elements of \mathcal{T}_d are related to **FPP** in the following way. Any $\mathbf{x} = (x_1, \dots, x_n) \in (k[t]_{\geq 0})^n$ such that $(\deg x_1, \dots, \deg x_n) \in \mathcal{T}_d$ yields a

solution to (1.1), for some monic polynomial F of degree d . Conversely, given a monic polynomial F of positive degree d , a solution $\mathbf{x} = (x_1, \dots, x_n) \in (k[t]_{\geq 0})^n$ of (1.1) yields the n -tuple $(\deg x_1, \dots, \deg x_n) \in \mathcal{T}_d$.

The strategy of our algorithm is to run through all integers d which are not larger than the upper bound given by Remark 2.4, and find the largest d for which $\mathcal{F}_d \subsetneq \mathcal{M}_d$. In order to follow this strategy, we need to find criteria to decide whether or not $\mathcal{F}_d = \mathcal{M}_d$. In this section, we give such a criteria. It is based on the fact that \mathcal{F}_d is a finite union of affine subspaces of \mathcal{P}_d .

Definition 6.2. Let \mathcal{V} be a finite dimensional k -vector space. We say that \mathcal{A} is an *affine subspace* of \mathcal{V} if there exist a vector subspace $\mathcal{U} \subset \mathcal{V}$ and a vector $\mathbf{v} \in \mathcal{V}$ such that $\mathcal{A} = \mathcal{U} + \mathbf{v}$. The *dimension* of \mathcal{A} , $\dim \mathcal{A}$, is defined to be $\dim \mathcal{U}$.

Remark 6.3. In the sequence, we use the following easy facts about affine subspaces whose proofs are left to the reader.

- (1) Let \mathcal{A} and \mathcal{B} be affine subspaces with $\dim \mathcal{A} = \dim \mathcal{B}$. If $\mathcal{A} \subset \mathcal{B}$ then $\mathcal{A} = \mathcal{B}$.
- (2) If $\mathbf{u}, \mathbf{v} \in \mathcal{A}$ and $\alpha \in k$ then $(1 - \alpha)\mathbf{u} + \alpha\mathbf{v}$ is also an element of \mathcal{A} .
- (3) Notice that inside k^{d+1} , the set \mathcal{M}_d of monic polynomials of degree d is the affine subspace $(1, 0, \dots, 0) + (0, \psi_d, \psi_{d-1}, \dots, \psi_0)$, and $\dim \mathcal{M}_d = d$.

For each $T \in \mathcal{T}_d$, below we define a matrix A_T and a column matrix B_T , both with $d + 1$ rows. Ultimately, we associate to each T the affine space given by the translation of the column space of A_T by the vector B_T . In case $\mathcal{R}_T = \emptyset$, we define A_T and B_T to be the zero matrix of order $(d + 1) \times 1$ and $\sum_{i \in \mathcal{S}_T} D_i$, respectively. Otherwise, for $i \in \mathcal{R}_T$, we first define the following $(d + 1) \times (e_i + 1)$ matrix

$$M_i = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ \alpha_{i(a_i-1)} & 1 & 0 & \cdots & 0 \\ \alpha_{i(a_i-2)} & \alpha_{i(a_i-1)} & 1 & \ddots & \vdots \\ \vdots & \alpha_{i(a_i-2)} & \alpha_{i(a_i-1)} & \ddots & 0 \\ \alpha_{i1} & \vdots & \alpha_{i(a_i-2)} & \ddots & 1 \\ \alpha_{i0} & \alpha_{i1} & \vdots & \ddots & \alpha_{i(a_i-1)} \\ 0 & \alpha_{i0} & \alpha_{i1} & \vdots & \alpha_{i(a_i-2)} \\ 0 & 0 & \alpha_{i0} & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \alpha_{i1} \\ 0 & 0 & 0 & \cdots & \alpha_{i0} \end{bmatrix}$$

where the j -th column of M_i is the vector representation in k^{d+1} of the polynomial $A_i t^{e_i-j+1}$, for $1 \leq j \leq e_i + 1$. Let \bar{M}_i be the $(d+1) \times e_i$ matrix obtained from M_i by removing its first column C_i . We define A_T to be the block row matrix of order $(d+1) \times \text{ind}(T)$

$$A_T = [\bar{M}_i]_{i \in \mathcal{R}_T}$$

and B_T to be the column matrix of order $(d+1) \times 1$

$$B_T = \sum_{i \in \mathcal{R}_T} C_i + \sum_{i \in \mathcal{S}_T} D_i,$$

where $\sum_{i \in \mathcal{S}_T} D_i$ is defined to be the zero vector, if $\mathcal{S}_T = \emptyset$.

- Remark 6.4.** (1) If $x_i = \sum_{j=0}^{e_i} \chi_{ij} t^j$ is a polynomial with $\deg x_i = e_i$, then the product of polynomials $x_i A_i$ is an element of \mathcal{P}_d and can be identified under (6.1) with the product of matrices $M_i X_i$, where X_i is the column vector $(\chi_{ie_i}, \dots, \chi_{i0})$.
- (2) We have that $\text{rank } A_T \leq d$. This is obvious if $\mathcal{R}_T = \emptyset$. If $\mathcal{R}_T \neq \emptyset$, first notice that the highest possible rank for the matrices M_i can only happen when the first entry in C_i is 1. Even in this case, if we remove the first column C_i of M_i , we are left with the matrix \bar{M}_i whose first row is zero. Consequently, A_T has at most d non-zero rows.
- (3) For any $T \in \mathcal{T}_d$, the first entry in B_T is 1. Therefore, if \mathcal{V}_T is the column space of the matrix A_T , then $\mathcal{V}_T + B_T \subset \mathcal{M}_d$ under the identification given by (6.1).

We are now ready to prove the first main result of this section.

Theorem 6.5. *Let d , \mathcal{F}_d , A_T and B_T be defined as above. Then \mathcal{F}_d is the union of a finite number of affine subspaces of \mathcal{P}_d . More precisely, under the identification of \mathcal{P}_d with k^{d+1} ,*

$$\mathcal{F}_d = \bigcup_{T \in \mathcal{T}_d} (\mathcal{V}_T + B_T),$$

where \mathcal{V}_T is the column space of the matrix A_T .

Proof. All we need to do is to prove the equality $\mathcal{F}_d = \bigcup_{T \in \mathcal{T}_d} (\mathcal{V}_T + B_T)$.

Let $T = (e_1, \dots, e_n) \in \mathcal{T}_d$, and let \mathcal{R}_T and \mathcal{S}_T be defined as in (6.2). For such T , we construct an n -tuple $(x_1, \dots, x_n) \in (k[t]_{\geq 0})^n$ such that $T = (\deg x_1, \dots, \deg x_n)$. First, we let $x_i = 1$ or $x_i = 0$, if $i \in \mathcal{S}$ or $i \notin \mathcal{R}_T \cup \mathcal{S}_T$, respectively. Otherwise $i \in \mathcal{R}_T$, and we can choose x_i to be any monic polynomial of degree e_i

$$x_i = t^{e_i} + \sum_{j=0}^{e_i-1} \chi_{ij} t^j.$$

Consider the $(e_i + 1) \times 1$ matrix

$$X_i = \begin{bmatrix} 1 \\ \chi_{i(e_i-1)} \\ \vdots \\ \chi_{i0} \end{bmatrix},$$

and let \bar{X}_i be the $e_i \times 1$ matrix obtained from X_i by removing its first row. By definition of \mathcal{T}_d , the product $x_i A_i$ is a polynomial of degree $\leq d$, which we identify with a $(d+1) \times 1$ column matrix, as in (6.1). The column matrix $x_i A_i$ is the zero matrix if $i \notin \mathcal{R}_T \cup \mathcal{S}_T$; it is the matrix D_i , if $i \in \mathcal{S}_T$; and if $i \in \mathcal{R}_T$, it is equal to the product of matrices $M_i X_i$.

In case $\mathcal{R}_T \neq \emptyset$, the above discussion shows that $\sum_{i=1}^n x_i A_i$, when identified with the $(d+1) \times 1$ column matrix in (6.1), satisfies

$$\begin{aligned} \sum_{i=1}^n x_i A_i &= \sum_{i \in \mathcal{R}} M_i X_i + \sum_{i \in \mathcal{S}} D_i \\ &= \sum_{i \in \mathcal{R}} C_i + \sum_{i \in \mathcal{R}} \bar{M}_i \bar{X}_i + \sum_{i \in \mathcal{S}} D_i \\ &= \sum_{i \in \mathcal{R}} \bar{M}_i \bar{X}_i + B_T \end{aligned}$$

It follows from basic properties of matrices that $\sum_{i \in \mathcal{R}} \bar{M}_i \bar{X}_i$ is a linear combination of the columns of the matrix A_T . Therefore, under the identification in (6.1), the set of linear combinations $\sum_{i=1}^n x_i A_i$ such that $(x_1, \dots, x_n) \in (k[t]_{\geq 0})^n$, $(\deg x_1, \dots, \deg x_n) \in \mathcal{T}_d$ and $\mathcal{R}_{(\deg x_1, \dots, \deg x_n)} \neq \emptyset$ is equal to

$$\bigcup_{\substack{T \in \mathcal{T}_d \\ \mathcal{R}_T \neq \emptyset}} (\mathcal{V}_T + B_T).$$

When $\mathcal{R}_T = \emptyset$, then

$$\sum_{i=1}^n x_i A_i = \sum_{i \in \mathcal{S}} D_i = A_T + B_T,$$

which shows that, in any case, the set of linear combinations $\sum_{i=1}^n x_i A_i$ such that $(x_1, \dots, x_n) \in (k[t]_{\geq 0})^n$ and $(\deg x_1, \dots, \deg x_n) \in \mathcal{T}_d$ is equal to $\bigcup_{T \in \mathcal{T}_d} (\mathcal{V}_T + B_T)$.

On the other hand, from the definition of \mathcal{T}_d , it follows that the set of linear combinations $\sum_{i=1}^n x_i A_i$ such that $(x_1, \dots, x_n) \in (k[t]_{\geq 0})^n$ and $(\deg x_1, \dots, \deg x_n) \in \mathcal{T}_d$ is equal to \mathcal{F}_d . Therefore, $\mathcal{F}_d = \bigcup_{T \in \mathcal{T}_d} (\mathcal{V}_T + B_T)$ as desired. \square

The second main result of this section gives a criteria to decide whether $\mathcal{F}_d = \mathcal{M}_d$. It is a consequence of the description of \mathcal{F}_d contained in the previous result and the fact that a vector space cannot be covered by a finite union of proper subspaces.

Lemma 6.6. *Let \mathcal{A} be an affine space over a field k , and let $\mathcal{U}_i \subset \mathcal{A}$ be proper affine subspaces, for i in an indexing set I . If $\mathcal{A} = \bigcup_{i \in I} \mathcal{U}_i$ then $|I| \geq |k| + 1$.*

Proof. See for instance [Cla12, Section 3]. \square

Theorem 6.7. *Let d , \mathcal{F}_d , and A_T be defined as above. Suppose the base field k satisfies $|\mathcal{T}_d| < |k|$. Then $\mathcal{F}_d = \mathcal{M}_d$ if and only if $\text{rank } A_T = d$, for some $T \in \mathcal{T}_d$.*

Proof. As before, we identify \mathcal{P}_d with k^{d+1} using (6.1). Note that from Theorem 6.5, $\mathcal{F}_d = \mathcal{M}_d$ implies that \mathcal{M}_d is a finite union of proper affine subspaces. Therefore, the result we want to prove is essentially an application of Lemma 6.6. Nonetheless, below we provide a proof that follows that in [Cla12, Section 3] but which is more suitable for computations. Our ultimate goal is to use it to find a counter-example to FPP for A_1, \dots, A_n .

If $\text{rank } A_T = d$, for some $T \in \mathcal{T}_d$, then $\dim(\mathcal{V}_T + B_T) = d = \dim \mathcal{M}_d$ and $\mathcal{F}_d = \mathcal{M}_d$, since $\mathcal{V}_T + B_T \subset \mathcal{F}_d \subset \mathcal{M}_d$. To prove the converse, we show that if $\text{rank } A_T = \dim(\mathcal{V}_T + B_T) < d$ for all $T \in \mathcal{T}_d$, then $\bigcup_{T \in \mathcal{T}_d} (\mathcal{V}_T + B_T) \subsetneq \mathcal{M}_d$.

First, let \mathcal{V}_T^\perp be the orthogonal complement of \mathcal{V}_T under the canonical inner product $\mathbf{u} \cdot \mathbf{v}$ on k^{d+1} . If $\text{rank } A_T < d$ for all $T \in \mathcal{T}_d$, then $\dim \mathcal{V}_T^\perp \geq 2$. From Remark 6.4, it follows that $\mathbf{e} = (1, 0, \dots, 0) \in \mathcal{V}_T^\perp$. As a result, we can choose a non-zero vector $\mathbf{n}_T \in \mathcal{V}_T^\perp$ which is linearly independent from \mathbf{e} . Thus, $\mathcal{V}_T + B_T$ is a subset of

$$\mathcal{A}_T = \{\mathbf{u} \in k^{d+1} : \mathbf{e} \cdot \mathbf{u} = 1, \mathbf{n}_T \cdot \mathbf{u} = \mathbf{n}_T \cdot B_T\}.$$

Clearly, $\dim \mathcal{A}_T = d - 1$, for all $T \in \mathcal{T}_d$. Under these assumptions, it is enough to prove that $\bigcup_{T \in \mathcal{T}_d} \mathcal{A}_T \subsetneq \mathcal{M}_d$.

Without loss of generality, we assume that for all $U \in \mathcal{T}_d$

$$\mathcal{A}_U \setminus \bigcup_{T \in \mathcal{T}_d \setminus \{U\}} \mathcal{A}_T \neq \emptyset.$$

This guarantees the existence of a vector $\mathbf{u} \in \mathcal{M}_d$ such that $\mathbf{u} \in \mathcal{A}_U$ but $\mathbf{u} \notin \mathcal{A}_T$ for all $T \neq U$. Additionally, we can choose $\mathbf{v} \in \mathcal{M}_d \setminus \mathcal{A}_U$. We consider the line $\mathcal{D} = \{(1 - \alpha)\mathbf{u} + \alpha\mathbf{v} : \alpha \in k\} \subset \mathcal{M}_d$. The result follows if we are able to prove that $|\mathcal{A}_T \cap \mathcal{D}| \leq 1$, for all $T \in \mathcal{T}_d$. Indeed, in this case

$$\left| \mathcal{D} \cap \left(\bigcup_{T \in \mathcal{T}_d} \mathcal{A}_T \right) \right| = \left| \bigcup_{T \in \mathcal{T}_d} \mathcal{A}_T \cap \mathcal{D} \right| \leq |\mathcal{T}_d|.$$

Since $|k| = |\mathcal{D}|$, this proves that $\bigcup_{T \in \mathcal{T}_d} \mathcal{A}_T \subsetneq \mathcal{M}_d$ if $|k| > |\mathcal{T}_d|$.

To compute $\mathcal{A}_T \cap \mathcal{D}$, we need to solve for α the equation $\mathbf{n}_T \cdot [(1 - \alpha)\mathbf{u} + \alpha\mathbf{v}] = \mathbf{n}_T \cdot B_T$, which can be simplified into

$$[\mathbf{n}_T \cdot (\mathbf{v} - \mathbf{u})]\alpha = \mathbf{n}_T \cdot (\mathbf{u} - B_T).$$

The above equation in α has more than one solution if and only if

$$\mathbf{n}_T \cdot (\mathbf{v} - \mathbf{u}) = 0 \quad \text{and} \quad \mathbf{n}_T \cdot (\mathbf{u} - B_T) = 0,$$

which, in turn, happens if and only if

$$\mathbf{n}_T \cdot \mathbf{v} = \mathbf{n}_T \cdot B_T \quad \text{and} \quad \mathbf{n}_T \cdot \mathbf{u} = \mathbf{n}_T \cdot B_T.$$

This last equation is equivalent to the fact $\mathbf{u}, \mathbf{v} \in \mathcal{A}_T$. Since this contradicts the choice of \mathbf{u} and \mathbf{v} , we conclude that $\mathcal{A}_T \cap \mathcal{D}$ has at most one element. We can actually say a bit more: $\mathcal{A}_T \cap \mathcal{D} = \emptyset$, if $\mathbf{n}_T \cdot \mathbf{v} = \mathbf{n}_T \cdot \mathbf{u}$. Otherwise, $[(1 - \alpha)\mathbf{u} + \alpha\mathbf{v}] \in \mathcal{A}_T$, for $\alpha = \mathbf{n}_T \cdot (\mathbf{u} - B_T) / [\mathbf{n}_T \cdot (\mathbf{v} - \mathbf{u})]$. \square

This last result allows us to prove the following lower bound for the Frobenius degree of A_1, \dots, A_n .

Corollary 6.8. *Suppose that the base field k satisfies $|\mathcal{T}_d| < |k|$. If d is an integer such that $\sum_{i=1}^n \max(d - a_i, 0) \leq d$ then $\mathcal{F}_d \subsetneq \mathcal{M}_d$. In particular,*

$$\max \left\{ d \in \mathbb{Z} : \sum_{i=1}^n \max(d - a_i, 0) \leq d \right\} \leq g(A_1, \dots, A_n).$$

Proof. Assume that d is an integer such that $\sum_{i=1}^n \max(d - a_i, 0) \leq d$. As a consequence of Theorem 6.7, to show that $\mathcal{F}_d \subsetneq \mathcal{M}_d$, we need to prove that $\text{rank } A_T < d$, for all $T \in \mathcal{T}_d$ with $\mathcal{R}_T \neq \emptyset$.

Since the number of columns of a matrix is an upper bound for its rank, it is true that

$$\text{rank } A_T \leq \text{ind}(T),$$

for all $T \in \mathcal{T}_d$ satisfying $\mathcal{R}_T \neq \emptyset$. If $\mathcal{R}_T = \{d - a_{j(T)}\}$, then $\text{rank } A_T \leq \text{ind}(T) = \sum_{i=1}^n \max(d - a_i, 0) = d - a_{j(T)} < d$. Otherwise $\mathcal{R}_T \neq \{d - a_{j(T)}\}$ and

$$\begin{aligned} \text{ind}(T) &\leq \sum_{i \in \mathcal{R}_T} e_i + \sum_{i \notin \mathcal{R}_T} \max(e_i, 0) \\ &< \sum_{i=1}^n \max(d - a_i, 0). \end{aligned}$$

Thus, if d is an integer such that $\sum_{i=1}^n \max(d - a_i, 0) \leq d$ then $\text{rank } A_T < d$, for all $T \in \mathcal{T}_d$ with $\mathcal{R}_T \neq \emptyset$. \square

Remark 6.9. For every $n \geq 2$, we can use Lemma 4.1 to show that the lower bound given in the previous corollary is sharp. In fact, choose pairwise coprime monic polynomials A_1, \dots, A_n such that $\deg A_i = a > 0$, for all $1 \leq i \leq n$. If $\tilde{A}_i = \prod_{j=1}^n A_j / A_i$, then $\deg \tilde{A}_i = (n - 1)a$ and, from Lemma 4.1,

$$g(\tilde{A}_1, \dots, \tilde{A}_n) = \deg A_1 + \dots + \deg A_n = na.$$

On the other hand na satisfies

$$\sum_{i=1}^n \max(na - \deg \tilde{A}_i, 0) = \sum_{i=1}^n \max(a, 0) \leq na.$$

Thus

$$na \leq \max \left\{ d \in \mathbb{Z} : \sum_{i=1}^n \max(d - \deg \tilde{A}_i, 0) \leq d \right\} \leq g(\tilde{A}_1, \dots, \tilde{A}_n) = na.$$

As discussed in Section 3.2, the Frobenius degree g of coprime monic polynomials A_1, \dots, A_n over k is not affected by a field extension K/k , if $|k|$ is sufficiently large. As we show below, this statement is also a consequence of Theorem 6.7.

Corollary 6.10. *Let A_1, \dots, A_n be coprime monic polynomials over a field k , let K/k be a field extension, and let g^+ be the upper bound given in Remark 2.4. If $|k| > |\mathcal{T}_{g^+}|$ then*

$$g_K(A_1, \dots, A_n) = g_k(A_1, \dots, A_n).$$

Proof. For a field extension K/k , we let $g_K = g_K(A_1, \dots, A_n)$. We want to show that $g_K = g_k$. First, we note that

$$g_k \leq g_K \leq g^+.$$

Thus we are left to prove $g_K \leq g_k$. For that matter, let d be a positive integer. It is not hard to see that \mathcal{T}_d depends only on d and the polynomials A_1, \dots, A_n ; and not on the base field in which the solutions of (1.1) are defined. Similarly, for all $T \in \mathcal{T}_d$, A_T and B_T are independent of the base field in which FPP is being considered. On the other hand, \mathcal{F}_d and \mathcal{M}_d depend on the base field K , and we make this dependence explicit by writing \mathcal{F}_d^K and \mathcal{M}_d^K , respectively. Since

$$|\mathcal{T}_{g_K}| \leq |\mathcal{T}_{g^+}| < |k| \leq |K|,$$

and $\text{rank } A_T$ is independent of the field extension K/k , it follows from the definition of g_K and two applications of Theorem 6.7 that $\mathcal{F}_{g_K}^k \subsetneq \mathcal{M}_{g_K}^k$. Therefore, $g_K \leq g_k$ as desired. \square

6.2. The algorithm. We use the notation of the previous section.

The algorithm we describe here only works under the assumption that the base field k satisfies $|k| > |\mathcal{T}_{g^+}|$, where g^+ is the upper bound obtained in Remark 2.4. Under this assumption, we can run through all integers $d \leq g^+$ in decreasing order and use Theorem 6.7 to check whether $\mathcal{F}_d = \mathcal{M}_d$. The first value of d for which $\mathcal{F}_d \subsetneq \mathcal{M}_d$ is the Frobenius degree of A_1, \dots, A_n . In case k is finite and $|k| \leq |\mathcal{T}_{g^+}|$ then the above strategy works except for the use of Theorem 6.7 to decide whether $\mathcal{F}_d = \mathcal{M}_d$. Instead, we can check whether such equality holds by one of the following “brute force” methods. \mathcal{F}_d can be constructed by computing all possible linear combinations

$$\sum_{i=1}^n x_i A_i = F,$$

with $x_i \in k[t]_{\geq 0}$ and $\deg F = d$. Then $\mathcal{F}_d \subsetneq \mathcal{M}_d$ if and only if $|\mathcal{F}_d| < q^d = |\mathcal{M}_d|$. Alternatively, one can construct $\bigcup_{T \in \mathcal{T}_d} (\mathcal{V}_T + B_T) = \mathcal{F}_d$ and

check whether $|\bigcup_{T \in \mathcal{T}_d} (\mathcal{V}_T + B_T)| = q^d$. It is unclear which of these two methods to check $\mathcal{F}_d = \mathcal{M}_d$ is less computationally expensive if they were to be implemented.

If we assume that $|k| > |\mathcal{T}_{g^+}|$ then the algorithm we use is less expensive than any of the above brute force methods because when we consider the matrix A_T , we are simultaneously considering all solutions (x_1, \dots, x_n) of (1.1) with $T = (\deg x_1, \dots, \deg x_n) \in \mathcal{T}_d$. Also, unlike the computation of $|\bigcup_{T \in \mathcal{T}_d} (\mathcal{V}_T + B_T)|$, we do not have to solve the large number of system of linear equations that are associated to all possible intersections of the form $(\mathcal{V}_T + B_T) \cap (\mathcal{V}_U + B_U)$. Our algorithm has been implemented in Sage², and it performed well in all the cases we have tried. It is also easy to implement if one only wants to compute $g(A_1, \dots, A_n)$. If one also wants to find a counter-example to FPP for A_1, \dots, A_n , then there are some added complications. These are due to the unpacking of some of the theoretical aspects of the argument for Theorem 6.7. In what follows, we first give a pseudo-code to compute $g(A_1, \dots, A_n)$. Later, we give more details on how to implement the construction of a counter-example for FPP. In both cases, we assume that the reader is able to implement the following sub-routines:

- **UPPERBOUND**(A_1, \dots, A_n).
Calculate an upper bound g^+ based on Remark 2.4.
Input: A_1, \dots, A_n .
Output: g^+ .
- **LOWERBOUND**(A_1, \dots, A_n).
Calculate the lower bound g^- given in Corollary 6.8.
Input: A_1, \dots, A_n .
Output: g^- .
- **TYPES**(d, A_1, \dots, A_n).
Construct the set \mathcal{T}_d .
Input: d and A_1, \dots, A_n .
Output: The list of elements in \mathcal{T}_d .
- **TMATRICES**(T).
Construct the matrices A_T and B_T .
Input: An element T of \mathcal{T}_d .
Output: The matrices A_T and B_T .

The pseudo-code for the computation of $g(A_1, \dots, A_n)$ is Algorithm 1 below.

²See the accompanying Sage worksheet on the first author's [personal website](#).

algorithm 1 Calculate $g(A_1, \dots, A_n)$.

Input: A_1, \dots, A_n .**Output:** $g(A_1, \dots, A_n)$.**Require:** $\gcd(A_1, \dots, A_n) = 1$, $\deg A_i > 0$, $n < p$ and $|k| > |\mathcal{T}_d|$. $g^+ \leftarrow \text{UPPERBOUND}(A_1, \dots, A_n)$ $g^- \leftarrow \text{LOWERBOUND}(A_1, \dots, A_n)$ **for** $d \leftarrow g^+$ **to** g^- **do** $\mathcal{T}_d \leftarrow \text{TYPES}(d, A_1, \dots, A_n)$ **for** $T = (e_1, \dots, e_n)$ in \mathcal{T}_d **do****if** $\sum_i \max(e_i, 0) < d$ **then** \triangleright If condition holds then $\text{rank } A_T < d$
and the algorithm can move on the next T .**else** $A_T, B_T \leftarrow \text{TMATRICES}(T)$ **if** $\text{rank } A_T = d$ **then** $\triangleright d \neq g(A_1, \dots, A_n)$.Decrease d and restart the loop for d .**else****end if****end if****end for****return** d **end for**

To construct a counter-example to **FPP** for A_1, \dots, A_n , we first transform the following qualitative statements contained in the proof of Theorem 6.7 into statements that can be checked algorithmically (we follow the notation in the proof of Theorem 6.7):

- **Statement 1:** $\mathcal{V}_T + B_T$ is a subset of an affine space \mathcal{A}_T with $\dim \mathcal{A}_T = d - 1$.

To be able to construct \mathcal{A}_T explicitly, we need to construct a non-zero vector \mathbf{n}_T orthogonal to \mathcal{V}_T and linearly independent from $\mathbf{e} = (1, 0, \dots, 0)$. This can be done by considering \mathbf{n}_T to be any non-zero solution of the linear system $A_T^t \mathbf{x} = \mathbf{0}$ which is also not a multiple of \mathbf{e} . Therefore, \mathcal{A}_T is simply the solution set of the system of linear equations on \mathbf{u}

$$\begin{cases} \mathbf{e} \cdot \mathbf{u} &= 1 \\ \mathbf{n}_T \cdot \mathbf{u} &= \mathbf{n}_T \cdot B_T \end{cases}.$$

- **Statement 2:** We can find $U \in \mathcal{T}_d$ such that $\mathcal{A}_U \setminus \bigcup_{T \in \mathcal{T}_d \setminus \{U\}} \mathcal{A}_T \neq \emptyset$.

In order to construct such U , all we need to do is pick any $U \in \mathcal{T}_d$ and construct a list \mathcal{L} of all $T \in \mathcal{T}_d$ for which $\mathcal{A}_T \neq \mathcal{A}_U$. Indeed, this is a consequence of the following claim: if $\mathcal{A}_U \subset \bigcup_{T \in \mathcal{J}} \mathcal{A}_T$, for some non-empty $\mathcal{J} \subsetneq \mathcal{T}_d$, then $\mathcal{A}_U = \mathcal{A}_V$, for some $V \in \mathcal{J}$. To prove the claim, first notice that under the assumption $\mathcal{A}_U \subset \bigcup_{T \in \mathcal{J}} \mathcal{A}_T$,

we have

$$\mathcal{A}_U = \bigcup_{T \in \mathcal{J}} (\mathcal{A}_T \cap \mathcal{A}_U).$$

We observe that either $\mathcal{A}_T \cap \mathcal{A}_U = \emptyset$ or $\mathcal{A}_T \cap \mathcal{A}_U$ is an affine subspace of \mathcal{A}_U . Moreover, not all non-empty $\mathcal{A}_T \cap \mathcal{A}_U$ is a proper subspace of \mathcal{A}_U ; otherwise Lemma 6.6 would contradict the assumption $|\mathcal{T}_d| < |k|$. Therefore, there exists $V \in \mathcal{J}$ such that $\mathcal{A}_V \cap \mathcal{A}_U = \mathcal{A}_U$ and, since $\dim \mathcal{A}_U = \dim \mathcal{A}_T$ for all $T \in \mathcal{T}_d$, it follows that $\mathcal{A}_U = \mathcal{A}_V$.

The intersection $\mathcal{A}_U \cap \mathcal{A}_T$ can be represented by the system of linear equations on \mathbf{u}

$$\begin{cases} \mathbf{e} \cdot \mathbf{u} = 1 \\ \mathbf{n}_U \cdot \mathbf{u} = \mathbf{n}_U \cdot B_U \\ \mathbf{n}_T \cdot \mathbf{u} = \mathbf{n}_T \cdot B_T \end{cases}.$$

Since $\dim \mathcal{A}_U = \dim \mathcal{A}_T$, this system has rank < 3 if and only if $\mathcal{A}_U = \mathcal{A}_T$. This fact can be used to check whether $\mathcal{A}_u = \mathcal{A}_T$ and construct \mathcal{L} . Without loss of generality, we may replace $\mathcal{T}_d \leftarrow \mathcal{L} \cup \{U\}$.

- **Statement 3:** *We can find vectors $\mathbf{u}, \mathbf{v} \in \mathcal{M}_d$ such that $\mathbf{u} \in \mathcal{A}_U$ but $\mathbf{u} \notin \mathcal{A}_T$ for all $T \neq U$, and $\mathbf{v} \notin \mathcal{A}_U$.*

To construct \mathbf{u} , we can randomly select $\mathbf{u} \in \mathcal{A}_U$ until

$$0 \neq \prod_{T \in \mathcal{T}_d} [\mathbf{n}_T \cdot (\mathbf{u} - B_T)].$$

From our choice of U , this routine is guaranteed to stop. The vector \mathbf{v} can be chosen as a solution of

$$\mathbf{e} \cdot \mathbf{v} = 1 \quad \text{and} \quad \mathbf{n}_U \cdot \mathbf{v} = \mathbf{n}_U \cdot B_U + 1.$$

- **Statement 4:** $\bigcup_{T \in \mathcal{T}_d} \mathcal{A}_T \subsetneq \mathcal{M}_d$, if $|k| > |\mathcal{T}_d|$.

Let $\Gamma = \{\alpha_T : T \in \mathcal{T}_d\}$, where $\alpha_T = 0$, if $\mathbf{n}_T \cdot \mathbf{v} = \mathbf{n}_T \cdot \mathbf{u}$; otherwise, $\alpha_T = \mathbf{n}_T \cdot (\mathbf{u} - B_T) / [\mathbf{n}_T \cdot (\mathbf{v} - \mathbf{u})]$. Since $\alpha_U = 0$, it follows from an argument in the proof of Theorem 6.7 that

$$|\Gamma| = \left| \mathcal{D} \cap \left(\bigcup_{T \in \mathcal{T}_d} \mathcal{A}_T \right) \right| \leq |\mathcal{T}_d| < |k|.$$

Therefore, if we randomly select an element $\beta \in k \setminus \Gamma$, then $\mathbf{w} = (1 - \beta)\mathbf{u} + \beta\mathbf{v}$ is such that $\mathbf{w} \in \mathcal{M}_d \setminus \bigcup_{T \in \mathcal{T}_d} \mathcal{A}_T$.

We are ready to give a pseudo-code for the construction of a counter-example to FPP for A_1, \dots, A_n . It should be straightforward to implement it in parallel with Algorithm 1.

algorithm 2 Calculate a counter example to **FPP** for A_1, \dots, A_n .

Input: A_1, \dots, A_n .**Output:** A polynomial G with $\deg G = g(A_1, \dots, A_n)$ for which (1.1) has no solution in $k[t]_{\geq 0}$.**Require:** $\deg A_i > 0$, $n < p$ and $|k| > |\mathcal{T}_d|$.**Ensure:** $g(A_1, \dots, A_n) = d$
procedure NORMALVECTOR(A_T, B_T) ▷ Compute \mathbf{n}_T .
Solve $A_T^t \mathbf{x} = \mathbf{0}$ $\mathbf{n}_T \leftarrow$ Non-zero solution \mathbf{x} that is not a multiple of $(1, 0, \dots, 0)$.**return** $[\mathbf{n}_T, B_T]$.**end procedure** $\mathbf{e} \leftarrow (1, 0, \dots, 0)$ $\mathcal{T}_d \leftarrow \text{TYPES}(d, A_1, \dots, A_n)$ $\mathcal{L} \leftarrow \{\text{NORMALVECTOR}(\text{TMATRICES}(T)) : T \in \mathcal{T}_d\}$ Choose $U \in \mathcal{T}_d$. $\mathcal{N} \leftarrow \emptyset$ ▷ Compute the set of normal vectors \mathbf{n}_T without any redundancy with \mathbf{n}_U .**for** $[\mathbf{n}_T, B_T]$ in \mathcal{L} **do**
if $\text{rank}\{\mathbf{e} \cdot \mathbf{w} = 1 \wedge \mathbf{n}_T \cdot \mathbf{w} = \mathbf{n}_T \cdot B_T \wedge \mathbf{n}_U \cdot \mathbf{w} = \mathbf{n}_U \cdot B_U\} = 3$ **then**
 $\mathcal{N} \leftarrow \mathcal{N} \cup \{[\mathbf{n}_T, B_T]\}.$
end if**end for** $\mathbf{u} \leftarrow \text{RANDOMELEMENT}(\mathcal{A}_U)$ ▷ Construct $\mathbf{u} \in \mathcal{A}_U \setminus \mathcal{A}_T$.**while** $0 = \prod_{[\mathbf{n}_T, B_T] \in \mathcal{N}} [\mathbf{n}_T \cdot (\mathbf{u} - B_T)]$ **do** $\mathbf{u} \leftarrow \text{RANDOMELEMENT}(\mathcal{A}_U)$ **end while** $\mathbf{v} \leftarrow$ solution of $\mathbf{e} \cdot \mathbf{v} = 1 \wedge \mathbf{n}_U \cdot \mathbf{v} = \mathbf{n}_U \cdot B_U + 1$

▷ Construct

 $\mathbf{v} \in \mathcal{M}_d \setminus \mathcal{A}_U$. $\Gamma \leftarrow \emptyset$ ▷ Construct the set Γ .**for** $\mathbf{n}_T \in \mathcal{N}$ **do****if** $\mathbf{n}_T \cdot \mathbf{u} \neq \mathbf{n}_T \cdot \mathbf{v}$ **then** $\Gamma \leftarrow \Gamma \cup \{\mathbf{n}_T \cdot (\mathbf{u} - B_T) / [\mathbf{n}_T \cdot (\mathbf{v} - \mathbf{u})]\}$ **end if****end for** $\beta \leftarrow \text{RANDOMELEMENT}(k^*)$ **while** $\beta \in \Gamma$ **do** $\beta \leftarrow \text{RANDOMELEMENT}(k^*)$ **end while****return** The polynomial associated to $(1 - \beta)\mathbf{u} + \beta\mathbf{v}$.

7. MORE EXAMPLES

There are a few questions that naturally arise when one starts to think and experiment with **FPP**. In this section, we use our algorithm to provide answers to some of these questions. In all of the examples, we are assuming

that the base field is $k = \mathbb{Q}$. Because of the natural limitation of the problem to the condition $|k| > \mathcal{T}_{g+}$, the questions over a finite field k have either a trivial answer or cannot be checked using our algorithm.

Example 7.1. *Does $g(A_1, \dots, A_n)$ depend only on $\deg A_1, \dots, \deg A_n$?*

Initially, it seemed plausible that $g(A_1, \dots, A_n)$ was independent of the polynomials A_1, \dots, A_n and depended only on their degrees. This is what happens in dimension 2, in the examples described by Theorem 4.1 and also on multiple tests we ran with our algorithm. It is not a general feature of **FPP** though, as the following examples illustrate.

Take $A_i = (t + i)^2$. Then $g(A_{-1}, A_0, A_1) = 3$. On the other hand, all the polynomials $B_i = t^2 - i$ also have degree 2, but $g(B_{-1}, B_0, B_1) = 4$.

Example 7.2. *Is $g(A_1, \dots, A_n)$ always equal to the upper bound in Remark 2.4 or the lower bound in Corollary 6.8?*

This question was also inspired by the examples in Section 4 for which we were able to compute $g(A_1, \dots, A_n)$ exactly. This question has a negative answer for $A_i = (t + i)^7$, $i = -1, 0, 1$. Corollary 6.8 and Remark 2.4 implies that $10 \leq g(A_{-1}, A_0, A_1) \leq 14$, when, in fact, $g(A_{-1}, A_0, A_1) = 11$.

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